

Problem I-1

Determine all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x + yf(x + y)) = f(x)f(y) + y^2$$

for all $x, y \in \mathbb{R}$.

Problem I-2

The *mixture* $M(A)$ of a finite sequence $A = (a_1, a_2, \dots, a_n)$ is the set of sums of all the non-empty blocks of consecutive terms, i.e.,

$$M(A) = \{a_i + a_{i+1} + \dots + a_j \mid 1 \leq i \leq j \leq n\}.$$

For example, the mixture of the sequence $A = (2, 3, 2)$ is the set $M(A) = \{2, 3, 5, 7\}$.

For each $n \geq 1$, let $k(n)$ be the smallest possible value such that there exist two finite sequences $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$ of positive integers whose mixtures $M(A)$ and $M(B)$ are disjoint and

$$(a_1 + a_2 + \dots + a_n) + (b_1 + b_2 + \dots + b_n) = k(n).$$

Determine $k(n)$ for all $n \geq 1$.

Problem I-3

Let ABC be an acute triangle with $AB < AC$ and let O be its circumcentre. Points P and Q lie on the line through A perpendicular to BC and satisfy $PA = PB$ and $QA = QC$. Let M and N be the midpoints of PQ and BC , respectively. Let $R \neq A$ be the second intersection of line AN with the circumcircle of ABC . Prove that the perpendicular bisectors of the three line segments BC , MO , and NR intersect at one point.

Problem I-4

Let $a_1, a_2, a_3, \dots$ be an infinite sequence of positive rational numbers such that for all $n \geq 1$ we have $a_{n+1} = \frac{p_n+1}{q_n+1}$ where $\frac{p_n}{q_n}$ is the reduced fraction form of a_n . Determine all possible values of a_1 such that there exists a positive integer k for which the product $a_1 a_2 \dots a_k$ is an integer.

Remark: The reduced fraction form of a positive rational number r is the unique representation $r = \frac{p}{q}$ where p and q are coprime positive integers.